Instructions

- You're welcome to work on these problems individually or in groups.
- If you have any questions, please don't hesitate to raise your hand and call me over! I'll be happy to help.

The Symmetric Group

Q1. *[Warm-up.]* Write down all the permutations in the symmetric group S_3. Express your answer in *matrix form*. For example, the permutation of $\{1, 2, 3\}$ given by $1 \to 2$, $2 \to 3$ and $3 \to 1$ is represented in matrix form as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Q2. *[Multiplication of permutations.]* In S_3, let

$$e = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \quad s = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Determine the following permutations and express your answer in matrix form.

(a) se.

(b) es.

(c) s^2. (This is ss.)

(d) st.

(e) ts.

Q3. *[Cycle notation.]* The following are permutations in S_4 that have been expressed in matrix form. Express them in cycle notation.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix}$.

Q4. *[Inverses.]* Referring back to Q2, determine the following permutations. Express your answers in both matrix form **and** cycle notation.

(a) e^{-1}.

(b) s^{-1}.

(c) t^{-1}.

(d) $(st)^{-1}$. [*Question:* How, if at all, is $(st)^{-1}$ related to s^{-1} and t^{-1}?]

Q5. *[Commutators]* An important concept in the proof of the insolvability of the quintic is the notion of *commutators* of permutations. A commutator of two permutations $s, t \in S_n$ is a permutation of the form

$$t^{-1}s^{-1}ts.$$

Reading from right to left, we interpet this permutation as

perform s, then perform t, then unperform s, then unperform t.

Since the order of multiplication in S_n is important, the above is **not** the same as $s^{-1}t^{-1}ts$, which would result in the identity permutation.

(a) Let $s = (12)$ and $t = (13)$ in S_3. Find the commutator $t^{-1}s^{-1}ts$.

(b) **[Challenging!]** Show that (1 2 3 4 5) is a commutator of permutations in S_5. That is, find permutations s and t in S_5 such that

$$(1\ 2\ 3\ 4\ 5) = t^{-1}s^{-1}ts.$$

[**Hint:** There are several possible t and s. Try to find s given that $t = (2\ 3\ 5\ 1\ 4)$.]

Note: A key step in the proof of the insolvability of the quintic is to show that *every* 5-cycle permutation in S_5 (like (1 2 3 4 5)) is a commutator of two other permutations, which are themselves 5-cycles. So then every 5-cycle is a commutator of 5-cycles which are commutators of 5-cycles which are commutators of 5-cycles, etc. (Nothing like this happens in S_2, S_3 or S_4. There are no "infinitely long" commutators of commutators. This is part of the reason why we have quadratic, cubic and quartic formulas!)